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Spectral Properties of Bounded and Unbounded Linear Operators on Hilbert Spaces

Abstract : Spectral theory is a fundamental branch of functional analysis that investigates the spectrum of linear operators acting on Hilbert and Banach spaces. It provides a powerful mathematical framework for studying differential equations, quantum mechanics, mathematical physics, signal processing, and numerical analysis. The spectral behaviour of bounded and unbounded linear operators has attracted considerable attention because of its theoretical significance and wide range of applications in both pure and applied mathematics.

This paper presents a comprehensive study of the spectral properties of bounded and unbounded linear operators defined on complex Hilbert spaces. The work begins with the fundamental concepts of Hilbert spaces, linear operators, boundedness, closed operators, and densely defined operators, followed by a rigorous analysis of the spectrum and the resolvent set. The spectrum is classified into the point spectrum, continuous spectrum, and residual spectrum, and their mathematical characterristics are examined in detail. The relationship between operator properties and spectral behaviour is established through several important classes of operators, including self-adjoint, normal, unitary, and compact operators.

Special emphasis is placed on the spectral theory of unbounded operators, where the role of operator domains, closedness, and self-adjoint extensions becomes essential. The paper discusses the existence and properties of resolvent operators, the Spectral Theorem for bounded and unbounded self-adjoint operators, and the spectral decomposition of compact operators. Several mathematical examples and illustrative applications are included to demonstrate the practical interpretation of the theoretical concepts.

Furthermore, the applications of spectral theory in quantum mechanics, Sturm–Liouville problems, evolution equations, boundary value problems, and computational mathematics are explored. The analysis highlights how eigenvalues, eigenvectors, and spectral measures provide effective tools for solving infinite-dimensional mathematical models.

The results presented in this paper provide a unified theoretical framework for understanding the spectral structure of bounded and unbounded linear operators on Hilbert spaces. The study contributes to the development of modern operator theory and offers a foundation for future research in differential operators, semigroup theory, numerical spectral approximation, and mathematical physics.

Keywords : Spectral Theory; Hilbert Spaces; Functional Analysis; Operator Theory; Bounded Linear Operators; Unbounded Linear Operators; Spectrum; Resolvent Operator; Point Spectrum; Continuous Spectrum; Residual Spectrum; Self-Adjoint Operators; Normal Operators; Compact Operators; Closed Operators; Densely Defined Operators; Spectral Theorem; Eigenvalues; Eigenvectors; Quantum Mechanics; Differential Operators; Sturm–Liouville Problems; Evolution Equations; Boundary Value Problems; Mathematical Physics.

1. Introduction : Spectral theory is one of the fundamental branches of functional analysis concerned with the study of linear operators acting on infinite-dimensional vector spaces. It generalizes the classical theory of eigenvalues and eigenvectors of matrices to bounded and unbounded linear operators on Hilbert and Banach spaces. Spectral theory has become an indispensable mathematical tool in operator theory, differential equations, quantum mechanics, mathematical physics, signal processing, and numerical analysis because it provides a unified framework for understanding the structural properties of linear transformations [1,2].

Let (H) be a complex Hilbert space equipped with the inner product

Let

$$T: D(T) \subseteq H \longrightarrow H$$

be a linear operator defined on a Hilbert space H. The spectrum of T, denoted by $\sigma(T)$, is defined as

$$\sigma(T) = \{\lambda \in \mathbb{C}: (T - \lambda I) \text{ is not invertible}\}.$$

Unlike finite-dimensional matrices, where the spectrum consists only of eigenvalues, the spectrum of an infinite-dimensional linear operator is generally decomposed into three mutually disjoint parts,

$$\sigma(T) = \sigma_p(T) \cup \sigma_c(T) \cup \sigma_r(T),$$

where

$$\sigma_p(T) = \{\lambda \in \mathbb{C}: (T - \lambda I)x = 0,\ x \neq 0\},$$

is called the point spectrum,

$$\sigma_c(T) = \left\{\lambda \in \mathbb{C}: (T - \lambda I) \text{ is one-to-one, } \overline{R(T - \lambda I)} = H, \text{ but } (T - \lambda I)^{-1} \text{ is unbounded}\right\},$$

is the continuous spectrum, and

$$\sigma_r(T) = \left\{\lambda \in \mathbb{C}: \overline{R(T - \lambda I)} \neq H\right\},$$

is called the residual spectrum.

A linear operator

$$T: H \to H$$

is said to be bounded if there exists a constant $M > 0$such that

$$\| Tx \| \leq M \| x \|, \forall x \in H.$$

The operator norm of $T$is defined by

$$\| T \| = \sup_{\|x\|=1} \| Tx \|.$$

One of the fundamental results of spectral theory states that every bounded linear operator on a complex Hilbert space has a non-empty compact spectrum satisfying

$$\sigma(T) \subseteq \{\lambda \in \mathbb{C}: |\lambda| \leq \| T \|\}.$$

Furthermore, the spectral radius of $T$is defined by

$$r(T) = \sup\{|\lambda| : \lambda \in \sigma(T)\},$$

and is given by the Spectral Radius Formula

$$r(T) = \lim_{n\to\infty} \| T^n \|^{\frac{1}{n}}.$$

This formula establishes a fundamental relationship between the spectrum of a bounded operator and its operator norm, and it plays a central role in modern operator theory.

The completeness property of (H) implies that every Cauchy sequence converges with respect to the induced norm. Consequently, Hilbert spaces provide the natural setting for studying infinite-dimensional operator equations and spectral decomposition [2].

Consider a linear operator be a linear operator, where $D(T)$denotes the domain of the operator T.

The operator $T$is said to be linear if

$$T(\alpha x + \beta y) = \alpha T(x) + \beta T(y),$$

for all

$$x, y \in D(T), \alpha, \beta \in \mathbb{C}.$$

If there exists a constant

$$M > 0$$

such that

$$\| Tx \| \leq M \| x \|, \forall x \in H,$$

then $T$is called a bounded linear operator.

The norm of a bounded linear operator T, called the operator norm, is defined by

$$\| T \| = \sup_{\|x\|=1} \| Tx \|.$$

Equivalently,

$$\| T \| = \sup_{x\neq 0} \frac{\| Tx \|}{\| x \|}.$$

Hence, every bounded operator satisfies

$$\| Tx \| \leq \| T \| \ \| x \|, \forall x \in H.$$

This is the standard mathematical notation used in functional analysis and operator theory.

Every bounded linear operator is continuous on the whole Hilbert space, whereas many important differential operators are naturally unbounded and require careful specification of their domains [3].

The central object of spectral theory is the spectrum of a linear operator. For a linear operator (T), the spectrum is defined by

The spectrum of a linear operator $T$is defined by

$$\sigma(T) = \{\lambda \in \mathbb{C} : (T - \lambda I) \text{ is not invertible}\},$$

where Idenotes the identity operator on the Hilbert space H.

The complement of the spectrum is called the resolvent set,

$$\rho(T) = \mathbb{C} \setminus \sigma(T).$$

For every

$$\lambda \in \rho(T),$$

the corresponding resolvent operator is defined as

$$R(\lambda, T) = (T - \lambda I)^{-1}.$$

The spectrum of Tcan be decomposed into three mutually disjoint subsets:

$$\sigma(T) = \sigma_p(T) \cup \sigma_c(T) \cup \sigma_r(T),$$

where the point spectrum is given by

$$\sigma_p(T) = \{\lambda \in \mathbb{C} : (T - \lambda I)x = 0, x \neq 0\},$$

the continuous spectrum is

$$\sigma_c(T) = \left\{\lambda \in \mathbb{C} : (T - \lambda I) \text{ is injective}, \overline{R(T - \lambda I)} = H, (T - \lambda I)^{-1} \text{ is unbounded}\right\},$$

and the residual spectrum is

$$\sigma_r(T) = \left\{\lambda \in \mathbb{C} : \overline{R(T - \lambda I)} \neq H\right\}.$$

For bounded linear operators, an important result states that the spectrum is always non-empty and compact. Moreover,

$$\sigma(T) \subseteq \{\lambda \in \mathbb{C} : |\lambda| \leq \| T \|\}.$$

The spectral radius of $T$is defined as

$$r(T) = \sup\{|\lambda| : \lambda \in \sigma(T)\},$$

and satisfies the spectral radius formula

$$r(T) = \lim_{n\to\infty} \| T^n \|^{\frac{1}{n}}.$$

This establishes a fundamental relationship between the operator norm and spectral behaviour.
For unbounded operators arising from differential equations, a typical example is the second-order differential operator

$$L = -\frac{d^2}{dx^2},$$

defined on the domain

$$D(L) = \{u \in H^2(0,\pi) : u(0) = u(\pi) = 0\}.$$

The corresponding eigenvalue problem is

$$-\frac{d^2u}{dx^2} = \lambda u.$$

The solutions are given by

$$u_n(x) = \sin(nx),$$

with corresponding eigenvalues

$$\lambda_n = n^2, n = 1,2,3, \dots .$$

These eigenfunctions form an orthogonal basis of $L^2(0,\pi)$and provide the foundation of Fourier series expansion and Sturm–Liouville theory.
A self-adjoint operator satisfies

$$T = T^*,$$

which is equivalent to

$$\langle Tx, y\rangle = \langle x, Ty\rangle, x, y \in D(T).$$

For a self-adjoint operator,

$$\sigma(T) \subseteq \mathbb{R}.$$

The Spectral Theorem gives the representation

$$T = \int_{\sigma(T)} \lambda \, dE(\lambda),$$

where

$$E(\lambda)$$

is a projection-valued measure.

This theorem forms the mathematical basis of quantum mechanics and functional calculus.The main objective of this paper is to study the spectral properties of bounded and unbounded linear operators on Hilbert spaces. The analysis focuses on

$$\sigma(T), \rho(T), R(\lambda, T),$$

compact operators, self-adjoint operators, and the Spectral Theorem. Applications to quantum mechanics, evolution equations, Sturm–Liouville problems, and boundary value problems demonstrate the importance of spectral theory in modern mathematics and mathematical physics.

2. Literature Review : The development of spectral theory has played a fundamental role in the evolution of modern functional analysis and operator theory. The concept of spectrum was introduced as a generalization of eigenvalues of matrices to linear operators acting on infinite-dimensional spaces. Early contributions by Hilbert, Riesz, and von Neumann established the foundation of operator theory and provided the mathematical framework for studying spectral properties of operators in Hilbert spaces [1,2].

The classical theory of eigenvalues for finite-dimensional matrices considers the equation

$$Ax = \lambda x,$$

where A is a matrix, $x \neq 0$ is an eigenvector, and $\lambda$is the corresponding eigenvalue. This concept was generalized to linear operators

$$T: D(T) \subseteq H \rightarrow H,$$

where $H$is a Hilbert space. The spectrum of an operator $T$is defined as

$$\sigma(T) = \{\lambda \in \mathbb{C}: (T - \lambda I) \text{ is not invertible}\}.$$

This generalization allowed researchers to investigate operators whose behavior cannot be described only through eigenvalues [3].

Riesz developed the theory of compact operators, which became one of the most important areas of spectral analysis. A compact operator

$$T: H \rightarrow H$$

maps every bounded sequence

$$\{x_n\}_{n=1}^{\infty}$$

into a sequence

$$\{Tx_n\}_{n=1}^{\infty}$$

having a convergent subsequence. For compact operators, every non-zero spectral value is an eigenvalue of finite multiplicity, and the only possible accumulation point of the spectrum is zero [4].

The spectral theory of bounded operators was extensively developed in the works of Banach and Gelfand. For a bounded linear operator satisfying

$$\| Tx \| \leq M \| x \|,$$

the spectrum satisfies

$$\sigma(T) \subseteq \{\lambda \in \mathbb{C}: |\lambda| \leq \| T \|\}.$$

Furthermore, the spectral radius formula

$$r(T) = \lim_{n \to \infty} \| T^n \|^{1/n}$$

establishes an important relationship between the spectrum and the norm of the operator [5].

A major breakthrough in spectral theory was achieved through the Spectral Theorem for self-adjoint operators. Von Neumann and Stone developed the spectral representation of self-adjoint operators satisfying

$$T = T^*.$$

For such operators, the spectrum is real,

$$\sigma(T) \subseteq \mathbb{R},$$

and the operator can be represented as

$$T = \int_{\sigma(T)} \lambda \, dE(\lambda),$$

where $E(\lambda)$is a projection-valued measure. This result became the foundation of quantum mechanics, where physical observables are represented by self-adjoint operators on Hilbert

spaces [2,6].
The study of unbounded operators was further developed by Kato, Reed, and Simon. Unlike bounded operators, an unbounded operator is defined on a proper dense subset

$$D(T) \subsetneq H.$$

For example, differential operators such as

$$L = -\frac{d^2}{dx^2}$$

play a central role in the analysis of partial differential equations and quantum mechanical systems. The domain of such operators determines important properties such as closedness, self-adjointness, and spectral structure [7].

The theory of closed and densely defined operators has provided a rigorous framework for studying unbounded operators. An operator $T$is closed if

$$x_n \to x,$$

and

$$Tx_n \to y$$

imply that

$$x \in D(T), Tx = y.$$

Closed operators are essential in the study of differential operators and evolution equations because many physically important operators are unbounded but closed [7,8].

The resolvent theory of operators has also been extensively investigated. The resolvent set is defined by

$$\rho(T) = \mathbb{C} \setminus \sigma(T),$$

and the resolvent operator is given by

$$R(\lambda, T) = (T - \lambda I)^{-1}.$$

The behavior of the resolvent provides important information about the stability of operator equations and plays a significant role in semigroup theory and perturbation analysis [8].

Recent developments in spectral theory focus on numerical spectral methods, spectral approximation, and applications to differential operators, quantum computing, and mathematical physics. Computational techniques such as finite element methods and spectral decomposition methods rely heavily on the theoretical framework developed for Hilbert space operators [9,10].

From the above studies, it is evident that spectral theory provides a unified approach for analyzing both bounded and unbounded operators. The existing literature establishes fundamental results concerning spectrum, resolvent operators, compactness, self-adjointness, and spectral decomposition. However, the relationship between the structural properties of operators and their spectral behavior continues to be an active area of research.

The present work aims to provide a systematic study of the spectral properties of bounded and unbounded linear operators on Hilbert spaces by combining classical results with modern applications in differential equations, quantum mechanics, and operator theory.

3. Preliminaries : In this section, we introduce the basic definitions and mathematical concepts required for the study of spectral properties of bounded and unbounded linear operators on Hilbert spaces. The fundamental notions of Hilbert spaces, linear operators, bounded operators, closed operators, adjoint operators, compact operators, and spectral concepts are presented.

3.1 Hilbert Spaces : A Hilbert space is a complete inner product space. Let Hbe a vector space over the field $\mathbb{C}$. An inner product is a mapping

$$\langle \cdot,\cdot \rangle : H \times H \to \mathbb{C},$$

which satisfies the following properties:

$$\langle x, y \rangle = \overline{\langle y, x \rangle},$$

$$\langle \alpha x + \beta y, z \rangle = \alpha \langle x, z \rangle + \beta \langle y, z \rangle,$$

and

$$\langle x, x \rangle \geq 0,$$

with

$$\langle x, x \rangle = 0 \iff x = 0.$$

The norm induced by the inner product is defined as

$$\| x \| = \sqrt{\langle x, x \rangle}, x \in H.$$

The space H is called a Hilbert space if every Cauchy sequence

$$\{x_n\}_{n=1}^{\infty}$$

satisfying

$$\| x_n - x_m \| \to 0, n, m \to \infty,$$

converges to an element $x \in H$.
Examples of Hilbert spaces include:

$$L^2(a, b) = \left\{ f : \int_a^b | f(x) |^2 \, dx < \infty \right\},$$

with inner product

$$\langle f, g \rangle = \int_a^b f(x) \, g(\bar{x}) \, dx,$$

and the sequence space

$$\ell^2 = \left\{ (x_1, x_2, \dots) : \sum_{n=1}^{\infty} | x_n |^2 < \infty \right\}.$$

3.2 Linear Operators : Let H be a Hilbert space. A mapping

$$T : D(T) \subseteq H \to H$$

is called a linear operator if

$$T(\alpha x + \beta y) = \alpha T(x) + \beta T(y),$$

for all

$$x, y \in D(T), \alpha, \beta \in \mathbb{C}.$$

The set

$$D(T)$$

is called the domain of the operator T, while

$$R(T) = \{ Tx : x \in D(T) \}$$

is called the range of T.

3.3 Bounded Linear Operators : A linear operator

$$T : H \to H$$

is called bounded if there exists a constant

$$M > 0$$

such that

$$\| Tx \| \leq M \| x \|, \forall x \in H.$$

The norm of the operator T is defined by

$$\| T \| = \sup_{\| x \| = 1} \| Tx \|.$$

Equivalently,

$$\| T \| = \sup_{x \neq 0} \frac{\| Tx \|}{\| x \|}.$$

The set of all bounded linear operators on H is denoted by

$$\mathcal{B}(H).$$

3.4 Unbounded Linear Operators : An operator

$$T: D(T) \subseteq H \to H$$

is called unbounded if no constant $M > 0$exists such that

$$\| Tx \| \leq M \| x \|, \forall x \in D(T).$$

A classical example is the differential operator

$$T = \frac{d}{dx},$$

defined by

$$Tf = f',$$

with domain

$$D(T) = H^1(0,1).$$

Such operators appear naturally in differential equations and mathematical physics.

3.5 Closed Operators : An operator

$$T: D(T) \subseteq H \to H$$

is called closed if its graph

$$G(T) = \{(x, Tx): x \in D(T)\}$$

is closed in

$$H \times H.$$

Equivalently, if

$$x_n \to x,$$

and

$$Tx_n \to y,$$

then

$$x \in D(T),$$

and

$$Tx = y.$$

Closed operators are essential in the theory of unbounded operators.

3.6 Dense Domain : The domain $D(T)$is called dense in Hif

$$D(\bar{T}) = H.$$

Most important unbounded operators in applications are densely defined.

3.7 Adjoint Operators : Let

$$T: D(T) \subseteq H \to H$$

be a densely defined operator. The adjoint operator

$$T^*$$

is defined by

$$\langle Tx, y \rangle = \langle x, T^* y \rangle,$$

for all

$$x \in D(T), y \in D(T^*).$$

The operator $T$is called self-adjoint if

$$T = T^*.$$

For self-adjoint operators,

$$\sigma(T) \subseteq \mathbb{R}.$$

3.8 Normal Operators : An operator $T$is called normal if

$$TT^* = T^*T.$$

Every self-adjoint operator is normal because

$$T = T^*$$

implies

$$TT^* = T^*T.$$

3.9 Compact Operators : A linear operator

$$T: H \to H$$

is compact if every bounded sequence

$$\{x_n\}$$

in Hhas a subsequence

$$\{x_{n_k}\}$$

such that

$$\{Tx_{n_k}\}$$

converges in H.

For compact operators,

$$\sigma(T) \setminus \{0\}$$

consists only of eigenvalues.

3.10 Spectrum and Resolvent : For a linear operator

$$T: D(T) \subseteq H \to H,$$

the spectrum is defined as

$$\sigma(T) = \{\lambda \in \mathbb{C}: (T - \lambda I) \text{ is not invertible}\}.$$

The resolvent set is

$$\rho(T) = \mathbb{C} \setminus \sigma(T).$$

For

$$\lambda \in \rho(T),$$

the resolvent operator is

$$R(\lambda, T) = (T - \lambda I)^{-1}.$$

3.11 Spectral Decomposition : For a compact self-adjoint operator T, the spectral decomposition is expressed as

$$Tx = \sum_{n=1}^{\infty} \lambda_n \langle x, e_n \rangle e_n,$$

where

$$\{e_n\}_{n=1}^{\infty}$$

is an orthonormal basis of eigenvectors and

$$\lambda_n$$

are the corresponding eigenvalues.

These preliminary concepts provide the mathematical foundation for analyzing the spectral properties of bounded and unbounded linear operators on Hilbert spaces. The following sections will discuss the spectrum, resolvent operators, spectral theorem, and applications in detail.

4. Spectral Properties of Bounded Operators : Let Hbe a complex Hilbert space and let

$$T \in \mathcal{B}(H)$$

be a bounded linear operator, where

$$T: H \to H.$$

The spectrum of $T$is defined as

$$\sigma(T) = \{\lambda \in \mathbb{C}: (T - \lambda I) \text{ is not invertible}\}.$$

For bounded operators, the spectrum has several important properties.

4.1 Non-Empty and Compact Spectrum : For every bounded linear operator Ton a complex

Banach space,

$$\sigma(T) \neq \emptyset.$$

Moreover, the spectrum is a compact subset of the complex plane and satisfies

$$\sigma(T) \subseteq \{\lambda \in \mathbb{C}: |\lambda| \leq \|T\|\}.$$

Hence,

$$|\lambda| \leq \|T\|, \forall \lambda \in \sigma(T).$$

4.2 Spectral Radius : The spectral radius of $T$is defined by

$$r(T) = \sup\{|\lambda|: \lambda \in \sigma(T)\}.$$

It satisfies the spectral radius formula

$$r(T) = \lim_{n\to\infty} \|T^n\|^{1/n}.$$

Therefore,

$$r(T) \leq \|T\|.$$

4.3 Resolvent of a Bounded Operator : The resolvent set is

$$\rho(T) = \mathbb{C} \setminus \sigma(T).$$

For

$$\lambda \in \rho(T),$$

the resolvent operator is

$$R(\lambda, T) = (T - \lambda I)^{-1}.$$

The resolvent satisfies the identity

$$R(\lambda, T) - R(\mu, T) = (\mu - \lambda)R(\lambda, T)R(\mu, T).$$

4.4 Point Spectrum of Bounded Operators : The point spectrum consists of eigenvalues satisfying

$$Tx = \lambda x, x \neq 0.$$

Equivalently,

$$(T - \lambda I)x = 0.$$

The set of eigenvalues is

$$\sigma_p(T) = \{\lambda \in \mathbb{C}: \ker(T - \lambda I) \neq \{0\}\}$$

4.5 Self-Adjoint Bounded Operators : If

$$T = T^*,$$

then $T$is self-adjoint. Its spectrum satisfies

$$\sigma(T) \subseteq \mathbb{R}.$$

For every $x \in H$,

$$\langle Tx, x\rangle \in \mathbb{R}.$$

The spectral decomposition is given by

$$T = \int_{\sigma(T)} \lambda\, dE(\lambda),$$

where $E(\lambda)$is the spectral measure.

4.6 Normal Operators : A bounded operator $T$is normal if

$$TT^* = T^*T.$$

For a normal operator,

$$\|T\| = r(T).$$

The Spectral Theorem gives

$$T = \int_{\sigma(T)} \lambda\, dE(\lambda).$$

Operators4.7 Compact Bounded : Let

$$T: H \to H$$

be compact. Then every non-zero spectral value is an eigenvalue:

$$\lambda \in \sigma(T), \lambda \neq 0$$

implies

$$Tx = \lambda x$$

for some

$$x \neq 0.$$

The eigenvalues satisfy

$$\lambda_n \to 0, n \to \infty.$$

For compact self-adjoint operators,

$$Tx = \sum_{n=1}^{\infty} \lambda_n \langle x, e_n \rangle e_n.$$

4.8 Example: Identity Operator : Consider

$$T = I.$$

Then

$$Tx = x.$$

The eigenvalue equation becomes

$$Tx = \lambda x,$$

so

$$x = \lambda x.$$

For $x \neq 0$,

$$\lambda = 1.$$

Hence,

$$\sigma(I) = \{1\}.$$

5. Spectral Theorem, Proofs, Examples, and Applications

5.1 Spectral Theorem for Bounded Self-Adjoint Operators : Let Hbe a complex Hilbert space and let

$$T \in \mathcal{B}(H)$$

be a bounded self-adjoint operator, i.e.,

$$T = T^*.$$

The Spectral Theorem states that there exists a unique projection-valued measure

$$E: \mathcal{B}(\sigma(T)) \to \mathcal{B}(H),$$

such that

$$T = \int_{\sigma(T)} \lambda \, dE(\lambda).$$

Here,

$$\sigma(T)$$

denotes the spectrum of T, and $E(\lambda)$represents the spectral projection associated with the spectral value λ.

The operator can also be expressed through the spectral resolution

$$T = \sum_{n=1}^{\infty} \lambda_n P_n,$$

for the case of a discrete spectrum, where

$$\lambda_n$$
are eigenvalues and
$$P_n$$
are orthogonal projection operators.

5.2 Proof of Spectral Theorem (Outline) : Let $T = T^*$. Since $T$is self-adjoint,
$$\langle Tx, x\rangle \in \mathbb{R},$$
for every
$$x \in H.$$
Define
$$m = \inf_{\|x\|=1} \langle Tx, x\rangle,$$
and
$$M = \sup_{\|x\|=1} \langle Tx, x\rangle.$$
Then,
$$\sigma(T) \subseteq [m, M].$$
Using the continuous functional calculus, for every continuous function
$$f: \sigma(T) \to \mathbb{C},$$
there exists an operator
$$f(T)$$
satisfying
$$f(T) = \int_{\sigma(T)} f(\lambda)\, dE(\lambda).$$
Choosing
$$f(\lambda) = \lambda,$$
we obtain
$$T = \int_{\sigma(T)} \lambda\, dE(\lambda).$$
Hence, the spectral representation of $T$is established.

5.3 Spectral Theorem for Compact Self-Adjoint Operators : Let
$$T: H \to H$$
be a compact self-adjoint operator. Then there exists an orthonormal basis
$$\{e_n\}_{n=1}^{\infty}$$
of eigenvectors such that
$$Te_n = \lambda_n e_n.$$
For any
$$x \in H,$$
we have the expansion
$$x = \sum_{n=1}^{\infty} \langle x, e_n\rangle e_n.$$
Applying T,
$$Tx = \sum_{n=1}^{\infty} \lambda_n \langle x, e_n\rangle e_n.$$
Therefore,

$$T = \sum_{n=1}^{\infty} \lambda_n P_n,$$

where

$$P_n x = \langle x, e_n \rangle e_n.$$

5.4 Example 1: Matrix Operator : Consider the operator

$$T = \begin{pmatrix} 2 & 0 \\ 0 & 3 \end{pmatrix}.$$

The eigenvalue equation is

$$Tx = \lambda x.$$

Therefore,

$$\det(T - \lambda I) = 0.$$

We obtain

$$\begin{vmatrix} 2-\lambda & 0 \\ 0 & 3-\lambda \end{vmatrix} = 0.$$

Hence,

$$(2-\lambda)(3-\lambda) = 0.$$

The eigenvalues are

$$\lambda_1 = 2, \lambda_2 = 3.$$

The spectral decomposition is

$$T = 2P_1 + 3P_2.$$

5.5 Example 2: Differential Operator : Consider the operator

$$L = -\frac{d^2}{dx^2}$$

on

$$D(L) = \{u \in H^2(0,\pi): u(0) = u(\pi) = 0\}.$$

The eigenvalue problem is

$$-\frac{d^2 u}{dx^2} = \lambda u.$$

The solution is

$$u_n(x) = \sin(nx),$$

with eigenvalues

$$\lambda_n = n^2, n = 1,2,3, \dots.$$

The spectral expansion becomes

$$Lu = \sum_{n=1}^{\infty} n^2 \langle u, e_n \rangle e_n.$$

This result forms the basis of Fourier analysis and Sturm–Liouville theory.

5.6 Applications of Spectral Theorem

(i) Quantum Mechanics : In quantum mechanics, physical observables are represented by self-adjoint operators.

For an observable A,

$$A = A^*.$$

The possible measurement values are given by

$$\sigma(A).$$

The state vector satisfies

$$A\psi = \lambda\psi.$$

The Hamiltonian operator is

$$H = -\frac{\hbar^2}{2m}\Delta + V(x),$$

and the Schrödinger equation is

$$H\psi = E\psi.$$

The energy levels are determined by the spectrum of H.

(ii) Differential Equations : Many boundary value problems can be written as

$$Lu = \lambda u.$$

The spectral decomposition provides solutions in the form

$$u(x) = \sum_{n=1}^{\infty} c_n\, \phi_n(x),$$

where

$$L\phi_n = \lambda_n \phi_n.$$

(iii) Evolution Equations : For an evolution equation

$$\frac{du}{dt} = Au,$$

the solution is expressed using the operator semigroup

$$u(t) = e^{tA}u(0).$$

Using spectral theory,

$$e^{tA} = \int_{\sigma(A)} e^{t\lambda}\ dE(\lambda).$$

(iv) Signal Processing : Spectral decomposition is used in Fourier analysis, where a signal is represented as

$$f(t) = \sum_{n=-\infty}^{\infty} c_n\, e^{int}.$$

The Fourier coefficients are

$$c_n = \frac{1}{2\pi}\int_{-\pi}^{\pi} f(t)\, e^{-i}\quad dt.$$

6. Conclusion : The spectral theory of linear operators on Hilbert spaces provides a fundamental framework for understanding the structure and behaviour of both bounded and unbounded operators. In this paper, the spectral properties of different classes of operators have been investigated, including bounded operators, compact operators, self-adjoint operators, normal operators, and closed densely defined unbounded operators.

For a bounded linear operator

$$T \in \mathcal{B}(H),$$

the spectrum

$$\sigma(T) = \{\lambda \in \mathbb{C} : (T - \lambda I) \text{ is not invertible}\}$$

is a non-empty compact subset of the complex plane and satisfies

$$\sigma(T) \subseteq \{\lambda \in \mathbb{C} : |\lambda| \leq \| T \|\}.$$

The spectral radius formula

$$r(T) = \lim_{n\to\infty} \| T^n \|^{1/n}$$

establishes an important relationship between the spectral properties and the norm of the operator.

The study of unbounded operators shows that the domain

$$D(T) \subseteq H$$

plays a crucial role in determining the spectral behaviour of differential operators. Operators such as

$$-\frac{d^2}{dx^2}$$

and the Schrödinger operator

$$H = -\frac{\hbar^2}{2m}\Delta + V(x)$$

demonstrate the importance of self-adjointness and closedness in obtaining meaningful spectral decompositions.

The Spectral Theorem provides a complete representation of self-adjoint operators in the form

$$T = \int_{\sigma(T)} \lambda \, dE(\lambda),$$

where $E(\lambda)$is a projection-valued measure. This result connects abstract operator theory with practical applications in quantum mechanics, differential equations, Fourier analysis, and mathematical physics.

Furthermore, compact operator theory shows that non-zero spectral values of compact operators correspond to eigenvalues with finite multiplicity, allowing infinite-dimensional problems to be treated similarly to finite-dimensional matrix problems.

The results discussed in this work establish that spectral analysis is an essential tool for investigating the structure of infinite-dimensional operators. Future research directions include spectral approximation methods, numerical computation of eigenvalues, perturbation theory, semigroup operators, and spectral analysis of nonlinear and quantum systems.

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