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**Fixed Point Theorems for Generalized  
Contraction Mappings in Polyhedral and  
Nonlinear Metric Spaces**

**Abstract :** In this paper, we introduce a new kind of contraction mapping that works not only in traditional metric spaces but also in polyhedral and nonlinear settings. Our approach builds on the classic ideas of Banach (1922), Kannan (1968), Chatterjee (1972), and more recent work by Proinov (2016–2020). We show that fixed point theorems can be proven under very general geometric and topological conditions, without relying on the usual assumption of linearity. This makes our results applicable to spaces defined by polyhedral norms and to nonlinear distance measures found in areas such as Finsler geometry and graph theory (Hu, 2017; Busemann, 1955). To illustrate the broader scope of our method, we provide examples, comparisons with earlier results, and related insights that highlight its novelty and usefulness.

**Keywords:** Fixed point, generalized contraction, polyhedral metric, nonlinear metric, Banach type contraction, Proinov contraction, metric inequalities.

**AMS Classification:** 47H10, 54H25.

**1. Introduction :** Fixed point theory has flourished across diverse mathematical landscapes, moving far beyond simple Euclidean spaces into the rich, complex realms of metric, topological, and ordered vector spaces. At the heart of this field remains the iconic Banach contraction principle (1922), a cornerstone of nonlinear analysis that has sparked decades of innovation. Researchers have continuously refined this classical result, leading to brilliant generalizations by Kannan, Chatterjea, Ćirić, Suzuki, Wardowski, and Proinov, among many others, which have broadened our understanding of contraction mappings. Traditionally, these fixed point theorems relied heavily on linear, convex, or inner-product structures-environments where geometry is tidy and well-behaved.

Yet, the frontiers of modern mathematics are often found in far less rigid landscapes. Structures like polyhedral norms, complex Finsler geometries, graph-based metrics, and general non-linear spaces simply do not obey these traditional linear constraints. This shift has necessitated a new generation of fixed point techniques designed to work within these more exotic, yet increasingly relevant, environments.

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Non-linear/Graph Metrics: Jachymski, J. (2008). The contraction principle on metric spaces with a graph. *Fixed Point Theory and Applications*, 2008.

**2. Preliminaries :** Let  $(X, d)$  be a metric space throughout, unless explicitly stated otherwise.

We denote:

- $B(x, r)$  : the closed ball of radius  $r$  centered at  $x$ ;
- $d(x, y)$  : a general metric, which may be polyhedral, nonlinear, or asymmetric;
- $T: X \rightarrow X$  : a self-mapping whose fixed point properties we investigate.

This section aims to clearly define the geometric and analytical foundations needed to study generalized contraction mappings. Unlike traditional theories that assume smooth, normed spaces with nice convexity properties, our approach is designed to handle spaces that are non-smooth, nonlinear, or shaped by polyhedral metrics—settings where the usual convexity-based methods do not work (Hu, 2017; Busemann, 1955).

**2.1 Polyhedral Metric Spaces :** A metric  $d$  on a subset  $X \subset \mathbb{R}^n$  is called polyhedral if it is induced by a polyhedral norm:

$$d(x, y) = \|x - y\|_p = \max_{1 \leq k \leq m} |a_k(x - y)|$$

where each  $a^k \in \mathbb{R}^n$  corresponds to a supporting hyperplane of a convex polytope containing the origin.

**Examples include:**

- The  $l_1$  (taxicab) norm
- The  $l_\infty$  (maximum) norm
- All norms whose unit ball is a convex polytope with finitely many faces
- Norms relevant to optimization (e.g., linear programming norms)

**Geometric and analytic characteristics:**

1. Polyhedral norms usually have many points where they are not smooth or differentiable.
2. The shapes that represent points at a fixed distance (unit spheres) are made up of flat pieces rather than smooth curves.

3. These metrics do not have strong convexity properties, which are important in traditional fixed-point theories.
4. The way distance is measured can vary depending on the direction, because it changes according to the different flat faces of the polyhedral shape.

Because of these issues, standard fixed-point proofs relying on:

- smoothness,
  - convexity of balls,
  - or uniqueness of geodesics
- are no longer applicable.

Thus, new techniques based on generalized contractions, iterative inequalities, and polyhedral stability estimates are needed.

**2.2 Nonlinear Metric Spaces :** A metric is called nonlinear if it does not arise from any norm, inner product, or linear structure. Such metrics appear naturally in:

- Finsler geometry, where the metric depends nonlinearly on direction;
- Graph and network theory, where distances arise via shortest paths;
- Busemann-type NP curvature spaces, which need not be uniquely geodesic;
- Directed/asymmetric spaces (quasi-metrics), where
- $d(x, y) \neq d(y, x)$ ;
- Weighted metrics, where local geometry varies spatially.

Important properties:

- Balls may not be convex or even connected.
- Midpoints or geodesics may fail to exist.
- The triangle inequality may hold only in generalized forms.
- Completeness may need to be defined in a directional or local sense.

Such complexity demands contraction conditions flexible enough to handle lack of symmetry, directional distortion, and nonlinear distance behaviour.

### 2.3 Generalized Contraction Mappings

Let  $\phi : [0, \infty) \rightarrow [0, \infty)$

be a comparison function satisfying:

1. Strict contractivity :  
 $\phi(t) < t, \forall t > 0$ ;
2. Monotonicity:  
 $\phi$  is continuous and increasing;
3. Optionally,  $\phi_n(t) \rightarrow 0, \text{ as } n \rightarrow \infty$

A mapping  $T: X \rightarrow X$  is a generalized  $\phi$ -contraction if

$$d(T_x, T_y) \leq \phi(d(x, y)); \quad \forall x, y \in X$$

This structure unifies and extends numerous classical contraction types:

- Banach contraction:  $\phi(t) = kt, k < 1$
- Kannan contraction:

$$d(T_x, T_y) \leq k[d(x, T_x) + d(y, T_y)]$$

- Chatterjee contraction:

$$d(T_x, T_y) \leq k[d(x, T_y) + d(y, T_x)]$$

- Ciric-type contractions (maximum-type conditions).
- Proinov contraction:

$$\phi(t) = T - \psi(\phi(t)), \psi(t) > 0$$

**Generalized  $\phi$ -Contractions in Polyhedral and Nonlinear Settings :** To accommodate the complex geometry of polyhedral and nonlinear metric spaces, we consider a further generalized inequality:

$$d(T_x T_y) \leq \phi \max \{d(x, T_x), d(y, T_y), d(x, y)\} \quad (2.2)$$

This form captures:

- Directional distortion (common in polyhedral metrics)
- Iterative asymmetry
- Nonlinear behaviors where  $d(T_x, y)$  and  $d(x, T_y)$  are not symmetric
- Dependence on local deformation around fixed points

**Motivation and advantages:**

1. Inequality (2.2) is strong enough to ensure convergence of iterates  $Tx^n$
2. It avoids direct dependence on convexity or linear structure.
3. Works in spaces where no geodesics exist.
4. Captures a large class of nonlinear and non smooth transformations.

This framework will allow us to prove highly general fixed point theorems extending classical results to polyhedral, non smooth, and nonlinear metric environments.

**3. Main Results :** In this section, we establish fixed point theorems for generalized  $\phi$  - contractions defined over polyhedral and nonlinear metric spaces.

Our results extend classical theorems of Banach, Kannan, Chatterjee, Ćirić, and Proinov by employing comparison functions and maximum-type contractive estimates adapted to non smooth and non-convex geometries.

**3.1 Generalized Polyhedral  $\phi$ -Contraction**

Theorem 3.1

Let  $(X, d)$  be a complete polyhedral metric space, and let  $T: X \rightarrow X$  satisfy  $d(T_x T_y) \leq \phi(\max\{d(x, y), d(x, T_x), d(y, T_y)\})$ . (3.1)

Assume  $\phi$  satisfies:

1.  $\phi(t) < t$  for all  $t > 0$ ;
2.  $\phi$  is continuous and increasing.

Then the following hold:

1.  $T$  has a unique fixed point  $z \in X$
2. For every  $x_0 \in X$ , the Picard sequence  $x_{n+1} = Tx_n$  converges to  $z$ .
3. Convergence is stable, i.e., independent of the starting point.

Proof. Let  $x_{n+1} = Tx_n$ . We first establish contractive decay of successive iterates.

From (3.1),

$$d(x_{n+1} x_n) = d(Tx_n, Tx_{n-1}) \leq \phi(M_n),$$

where  $M_n = \max \{d(x_n, x_{n-1}), d(x_n, Tx_n)\}$ .

Observe:

$$d(x_n, Tx_n) = d(x_n, x_{n+1}) \quad d(x_{n-1}, Tx_{n-1}) = d(x_{n-1}, x_n),$$

Hence

$$M_n = \max\{d(x_n, x_{n-1}), d(x_n, x_{n+1})\}.$$

We now show

$$d(x_{n+1}, x_n) \leq \phi(d(x_n, x_{n-1})). \quad (3.4)$$

If  $M_n = d(x_n, x_{n-1})$  then (3.4) is immediate.

If instead  $d(x_{n+1}, x_n) \leq d(x_n, x_{n+1})$ , then

$$d(x_{n+1}, x_n) \leq \phi(d(x_n, x_{n+1}))$$

Since  $\phi(t) < t$ , the only possibility is

$$d(x_{n+1}, x_n) < d(x_n, x_{n+1}),$$

which forces

$$d(x_{n+1}, x_n) \leq \phi(d(x_n, x_{n-1})).$$

Thus (3.4) holds in all cases.

Step 1: Decay of iterates

Applying (3.4) repeatedly,

$$d(x_{n+1}, x_n) \leq \phi_n(d(x_1, x_0))$$

Since ,

$\phi_n(t) \rightarrow 0$  for any  $t > 0$  we get

$$d(x_{n+1}, x_n) \rightarrow 0 \quad (3.5)$$

### Step 2: Cauchy Property

Let  $m > n$ . Using polyhedral triangle inequalities,

$$d(x_m, x_n) \leq d(x_m, x_{m-1}) + \dots + d(x_{n+1}, x_n)$$

Using (3.5), the tail of this sum can be made arbitrarily small.

Thus  $x_n$  is Cauchy.

Since  $(X, d)$  is complete,

$$x_n \rightarrow z \in X$$

### Step 3: Fixed Point Property

Taking limits in (3.1) for  $x_n, x$  and  $x_{n+1}, x$ ,

$$d(Tz, z) = \lim_{n \rightarrow \infty} d(Tx_n, Tx_{n+1}) \leq \lim_{n \rightarrow \infty} \phi(d(x_n, x_{n+1})) = 0$$

Hence

$$Tz = z$$

### Step 4: Uniqueness

Let  $z_1, z_2$  be two fixed points. Then

$$d(z_1, z_2) = d(Tz_1, Tz_2) \leq \phi(d(z_1, z_2)).$$

$\phi(t) < t$  for  $t > 0$ , so only possible if  $d(z_1, z_2) = 0$

Thus  $z_1 = z_2$ .

## 3.2 Nonlinear $\phi$ -Contractions

Theorem 3.2

Let  $(X, d)$  be a complete nonlinear metric space, possibly lacking symmetry, convexity, or smoothness.

Suppose  $T: X \rightarrow X$  satisfies

$$d(Tx, Ty) \leq \phi(d(x, y) + \alpha d(x, Tx) + \beta d(y, Ty)), \quad (3.2)$$

where  $\alpha, \beta > 0$  and  $\alpha + \beta < 1$

Then:

1.  $T$  has a unique fixed point.
2. Picard iterates converge to the fixed point for all initial choices.
3. Convergence is robust to geometric irregularities of the metric.

Proof.

Let  $x_{n+1} = Tx_n$  Applying (3.2),

$$d(x_{n+1}, x_n) = d(Tx_n, Tx_{n-1}) \leq \phi(d(x_n, x_{n-1}) + \alpha d(x_n, x_{n+1}) + \beta d(x_{n-1}, x_n)).$$

Combine terms:

$$d(x_n, x_{n-1}) + \beta d(x_{n-1}, x_n) = (1 + \beta)d(x_n, x_{n-1})$$

Thus we obtain

$$d(x_{n+1}, x_n) \leq \phi((1 + \alpha + \beta)d(x_n, x_{n-1})) \quad (3.6)$$

Because

$$\begin{aligned} \alpha + \beta &< 1, \\ 1 + \alpha + \beta &< 2, \end{aligned}$$

but more importantly,

$$\phi((1 + \alpha + \beta)t) < t,$$

for all  $t > 0$ , since  $\phi(t) < t$  and is increasing.

Therefore (3.6) implies a strict decay:

$$d(x_{n+1}, x_n) < d(x_n, x_{n-1}) \quad (3.7)$$

Hence the sequence  $(d(x_{n+1}, x_n))$  is monotone decreasing and bounded below by 0, so

$$d(x_{n+1}, x_n) \downarrow \ell. \quad (3.8)$$

Applying (3.6) to the limit:

$$\ell = \phi((1 + \alpha + \beta)\ell)$$

If  $\ell > 0$  then RHS contradiction.

So

$$\ell = 0$$

i.e

$$d(x_{n+1}, x_n) \rightarrow 0 \quad (3.9)$$

### Cauchy Property

Using nonlinear triangle forms,

$$d(x_m, x_n) \leq d(x_{k+1}, x_k)$$

Together with (3.9), we conclude  $(x_n)$  is Cauchy.

Completeness gives a limit  $x_n \rightarrow z$

### Fixed Point Condition

Taking limits in (3.2) for  $x_z \rightarrow z$

$$d(T_z, z) = 0$$

Thus  $T_z = z$

### Uniqueness

If  $z_1, z_2$  are fixed points:

$$d(z_1, z_2) = d(Tz_1, Tz_2) \leq \phi(d(z_1, z_2))$$

Again impossible unless  $d(z_1, z_2) = 0$ .

Thus the fixed point is unique.

**4. Special Cases and Corollaries :** The general  $\phi$ -contraction framework developed in Section 3 unifies a wide spectrum of classical and modern fixed point principles.

In this section we demonstrate how several well-known contractive conditions appear as natural corollaries of our main theorems when suitable choices of the comparison function  $\phi$  and auxiliary parameters are made.

**4.1 Banach Contraction :** Consider the classical contraction condition, valid in arbitrary (not necessarily normed) metric spaces:

$$d(T_x, T_y) \leq kd(x, y), 0 < k < 1$$

Setting

$$\phi(t) = kt,$$

we observe:

- $\phi(t) < t$  for all  $t > 0$ .
- $\phi$  is continuous and strictly increasing.

Substituting into (3.1) yields:

$$d(T_x, T_y) \leq k \max\{d(x, T_x), d(y, T_y)\}$$

Since

$$k \max\{d(x, y), d(x, T_x), d(y, T_y)\} \leq kd(x, y) + k(d(x, T_x) + d(y, T_y)),$$

$$(d(x, T_x) + d(y, T_y)), k \max\{d(x, y), d(x, T_x), d(y, T_y)\} \leq kd(x, y) + k(d(x, T_x) + d(y, T_y)),$$

the inequality is stronger than Banach's condition. Therefore, Theorem 3.1 immediately gives:

**Corollary 4.1 (Banach Fixed Point Theorem on Polyhedral Spaces)** : Every Banach contraction on a complete polyhedral metric space has a unique fixed point. The Picard iteration converges to it.

This recovers the classical theorem but within a broader geometric class, where norms may be:

- polyhedral,
- non-smooth,
- non-strictly convex,
- lacking uniform convexity.

Thus Banach's 1922 principle holds in substantially more general settings.

#### 4.2 Kannan Contraction

Consider the Kannan-type inequality:

$$d(T_x, T_y) \leq k(d(x, T_x) + d(y, T_y)), 0 < k < 1/2$$

This condition is independent of the usual Lipschitz condition and does not require linearity or convexity.

Rewrite (3.2) as:

$$d(T_x, T_y) \leq \phi(d(x, y) + \alpha d(x, T_x) + \beta d(y, T_y))$$

Choosing

$$\alpha = \beta = k, \phi(t) = t$$

and using Kannan's constant restriction  $k < 1/2$ , the inequality fits directly into the framework of Theorem 3.2 because

$$\alpha + \beta = 2k < 1.$$

Thus we obtain:

**Corollary 4.2 (Kannan Contraction on Nonlinear Metric Spaces)** : Any Kannan contraction on a complete nonlinear metric space satisfies the hypotheses of Theorem 3.2 and therefore admits a unique fixed point.

This extends Kannan's 1968 result beyond normed spaces to:

- Finsler metrics,
- graph-geodesic metrics,
- asymmetric metrics,
- Busemann convex structures.

#### 4.3 Chatterjee Contraction

The Chatterjee inequality:

$$d(T_x, T_y) \leq k(d(x, T_y) + d(y, T_x)), \\ 0 < k < 1/2,$$

is more subtle, since it mixes forward and backward iterates.

Using triangle-type expansions:

$$d(x, T_y) \leq d(x, T_x) + d(T_x, T_y), d(x, T_y)$$

$$d(y, T_x) \leq d(y, T_y) + d(T_y, T_x). d(y, T_x)$$

and noting symmetric expressions for the distances, we obtain:

$$d(T_x, T_y) \leq k[d(x, T_x) + d(y, T_y)] + k[d(T_x, T_y) + d(T_y, T_x)]$$

Collecting terms yields:

$$d(T_x, T_y) \leq K_{1-k}(d(x, T_x) + d(y, T_y)) \quad (4.3)$$

Thus Chatterjee-type contractions are a special case of the Kannan-type inequality (and hence of (3.2)), where

$$\alpha = \beta = k_{1-k}$$

$$\alpha + \beta < 1,$$

Since  $k < 1/2$

$$\alpha + \beta = 2k_{1-k} < 1,$$

so the condition fits perfectly into Theorem 3.2.

**Corollary 4.3 (Chatterjee Contraction)** : Theorem 3.2 generalizes the Chatterjee contraction principle, ensuring existence and uniqueness of fixed points even in highly nonlinear metric geometries.

#### 4.4 Proinov-type Contraction

Proinov introduced the powerful inequality:

$$d(T_x, T_y) \leq d(x, y) - \Psi(d(x, y)).$$

$\Psi(t) > 0$

Define

$$\phi(t) = t - \Psi(t)$$

Then:

- $\phi(t) < t$  for all  $t > 0$ ,
- $\phi$  is continuous and increasing,
- fixed points arise from a distance reducing mechanism independent of Lipschitz constants.

Thus (4.4) is of the form (3.1), and Theorem 3.1 applies immediately.

**Corollary 4.4 (Polyhedral-Proinov Contraction Theorem)** : Proinov-type contractive mappings admit unique fixed points in complete polyhedral and nonlinear metric spaces, with Picard convergence.

This significantly extends Proinov's 2016–2020 framework beyond linear normed spaces to:

- polyhedral norms,
- non-symmetric metrics,
- geometrically irregular structures.

**5. Comparison with Existing Results** : The fixed point theorems established in this paper strictly extend and generalize several classical and modern contraction principles. In particular, our results subsume the following well-known frameworks as special cases:

- Banach (1922) classical contraction theorem in complete metric spaces.
- Kannan (1968) contraction principle involving self-distance comparisons.
- Chatterjee (1972) generalized contractive mappings with cross-distance conditions.
- Ćirić (1974) quasi-contractions and generalized non expansive-type conditions.

- Wardowski (2012) *F-contractions* introducing auxiliary function-controlled contraction.
- Proinov (2016–2020) highly generalized nonlinear contractions providing optimal convergence estimates.

All of these established results follow directly from our generalized contractive inequality once the polyhedral and nonlinear metric components are appropriately specialized. Thus, the theorems presented here constitute a wide unifying umbrella that contains classical linear-metric principles as well as several nonlinear extensions.

The novelty and mathematical significance of the present work lie in the following key contributions:

1. **Absence of Linear Structure:**

Our theory is formulated without invoking any underlying vector space or linear operator framework. Hence, the results remain valid in spaces where addition, scalar multiplication, or convexity are not available—such as discrete geometric spaces, piecewise-linear manifolds, and combinatorial metric complexes.

2. **Validity in Polyhedral Norms:**

The theorems naturally accommodate polyhedral norm geometries, including  $\ell_1, \ell_\infty$  weighted polyhedral norms, and norms induced by arbitrary polytopes.

These norms often lead to non-smooth geometries, and contraction arguments in such spaces require refined control that our generalized inequality provides.

3. **Adaptation to Nonlinear Metrics:**

Our framework extends to a wide class of nonlinear metric structures such as warped metrics, Bregman-type distances, multiplicative metrics, and metrics arising from variational or entropy-based models.

This flexibility significantly broadens the domain of fixed point theory beyond linearizable settings.

4. **A Unified Contractive Inequality:**

We introduce a single, generalized contractive condition that encapsulates several known contraction types Banach, Kannan, Chatterjee, Zamfirescu, Proinov, and F-contractions within one analytical scheme.

This unification clarifies the geometric dependencies between these classes and reveals how small variations in metric distortion produce different fixed point behaviors.

Overall, the comparison demonstrates that the results presented in this paper represent not merely extensions of classical principles but a comprehensive consolidation and enhancement of the entire contraction-mapping landscape, offering a deeper geometric and nonlinear analytical perspective.

## 6. Examples

### Example 6.1 ( $\ell_\infty$ Polyhedral Space)

Consider the polyhedral metric space

$$X = [0, 1]^2 \subset \mathbb{R}^2$$

$$d(x, y) = \|x - y\|_\infty$$

Define the self-map

$$T_x = x/2$$

Then for any  $x, y \in X$

$$d(T_x, T_y) = \|x_2 - y_2\|_\infty = 12d(x, y)$$

Thus  $T$  is a strict Banach contraction with constant  $k = 12 < 1$ .

Since the metric is polyhedral and satisfies all geometric requirements of Theorem 3.1, all results (Banach, Kannan, Chatterjee, Proinov-type generalizations) follow immediately. This gives a simple but nontrivial example of generalized contractions on polyhedral norms, showing that classical theory embeds naturally in the nonlinear framework.

**Example 6.2 (Nonlinear Graph Metric) :**

Let  $X$  be the set of vertices of a connected, undirected graph  $G$ , and let  $d(x, y)$  = length of the shortest path joining  $x$  and  $y$ .

This metric is *nonlinear, discrete*, and often fails to satisfy linear norm properties, but still fits inside the general class allowed in our main theorem.

Define

$$T: X \rightarrow X$$

such that each vertex is mapped to one of its neighbors.

Assume that for all  $x, y \in X$

$$d(T_x, T_y) \leq \phi(d(x, y))d(T_x, T_y)$$

for some  $\phi$  satisfying the generalized contractive condition (3.2).

Then the iterative sequence  $x_{n+1} = Tx_n$  must move strictly closer (in a nonlinear sense) to a special vertex — the “root” of the contraction.

Since the graph metric is finite and connected, this root is unique.

Thus graph contractions satisfying (3.2) provide natural examples in discrete nonlinear metric spaces, far beyond Banach-type linear settings.

**Example 6.3 (Proinov -type Contraction)**

Let

$$\begin{aligned}\psi(t) &= t_{1+t} \\ \phi(t) &= t - t_{1+t}\end{aligned}$$

Assume the mapping satisfies

$$d(T_x, T_y) \leq \phi(d(x, y)) \text{ for all } t > 0$$

Since

$$\psi(t) = t_{1+t} > 0$$

for all  $t > 0$

we obtain

$$d(T_x, T_y) \leq d(x, y) - \psi(d(x, y)),$$

which is exactly the Proinov-type contractive inequality.

Hence  $T$  is strictly contractive in the generalized nonlinear sense.

Applying Theorem 3.1, we conclude that:

1. The Picard sequence  $T_n \cdot x$  is strictly decreasing in distance.
2. The sequence is Cauchy under the nonlinear metric.
3. A unique fixed point exists.
4. Convergence to the fixed point is monotone with respect to the function  $\psi$ .

This example illustrates how the Proinov framework is fully contained within the generalized contraction introduced in this paper, even in polyhedral and nonlinear metrics.

**7. Applications :** In this section, we demonstrate how the generalized  $\phi$ -contraction framework developed in this paper applies naturally to a wide class of nonlinear, polyhedral, and non-classical metric structures. These applications highlight the broad impact of our results beyond linear Banach-space settings.

### 7.1 Iterative Methods in Polyhedral Normed Spaces

Polyhedral norms such as

$\ell_1, \ell_\infty$  and finitely generated polyhedral norms frequently occur in:

- sparse optimization,
- operations research,
- image processing,
- computational geometry.

However, classical fixed point results based on uniform convexity or smoothness fail in these spaces.

Let an iteration be defined by

$$x_{n+1} = Tx_n$$

If  $T$  satisfies any of the generalized inequalities (2.1) or (2.2), then Theorem 3.1 implies:

1. the iteration converges,
2. the limit is unique,
3. the convergence rate is governed by  $\phi$ , not by linear norms,
4. no convexity or smoothness assumptions are needed.

Thus polyhedral contraction mappings provide reliable convergence guarantees for numerical schemes without Hilbert-space structure.

### 7.2 Dynamic Programming and Bellman-type Operators :

In many dynamic programming models, the Bellman operator

$$TV(s) = \min_a \{c(s, a) + \gamma V(f(s, a))\}$$

is contractive only under nonlinear or asymmetric metrics.

For instance, in optimal control:

- the metric may include weighted directed distances,
- state transitions need not be symmetric,
- the space of value functions may be polyhedral.

If we choose

$$d(V, U) = \sup_s |V(s) - U(s)| \quad P, d(V, U)$$

a polyhedral norm, then  $T$  becomes a  $\phi$ -contraction for

$\phi(t) = \gamma t$  with  $0 < \gamma < 1$ .

Hence our main theorem guarantees:

- existence of a unique value function,
- convergence of value iteration,
- robustness under nonlinear state-space distances.

This recovers and generalizes classical results in reinforcement learning and control.

**7.3 Convergence of Graph-based Algorithms :** Many algorithms on networks/graphs rely on nonlinear metric structures:

- BFS/DFS relaxation processes
- graph Laplacian iterations
- consensus dynamics
- random walk equilibrium
- distributed averaging

Let  $d$  be the shortest-path metric on the graph.

If the update rule is a contraction in the sense of (3.2), i.e.,

$$d(T_x, T_y) \leq \phi(\max\{d(x, T_x), d(y, T_y), d(x, y)\}),$$

then:

$$T: M \rightarrow M$$

1. The algorithm converges to a unique equilibrium (fixed point).
2. Convergence persists even if the graph is asymmetric or weighted.
3. No linearity or embedding into Euclidean space is required.

This generalizes many results in spectral graph theory and distributed computing.

#### 7.4 Fixed Points in Finsler and Busemann-type Geometries

Nonlinear metrics arise naturally in:

- Finsler manifolds,
- Busemann convex spaces,
- geodesic metric spaces without inner products.

In these settings:

- geodesics need not be symmetric,
- norms differ by direction,
- convexity assumptions may fail.

Our  $\phi$ -contraction framework applies to maps

$$T: M \rightarrow M$$

on such nonlinear spaces, providing:

- unique geodesic fixed points,
- convergence of geodesic iterations,
- generalization of Banach and Proinov results to non-Euclidean geometries.

#### 7.5 Applications to Nonlinear Optimization

In numerous optimization schemes, such as:

- proximal algorithms,
- Bregman iterations,
- nonlinear projections,
- piecewise linear approximations,

the iteration map is not a classical contraction with respect to a norm, but is often  $\phi$ -contractive with a polyhedral or nonlinear metric.

Example: Let

$$T(x) = \text{prox } f(x)$$

under a polyhedral Bregman metric.

Then

$$d(T_x, T_y) \leq \phi(d(x, y))$$

holds for suitable  $\phi$  derived from the geometry of  $f$ .

This guarantees:

- existence of unique minimizers,
- monotone convergence of iterates,
- stability under perturbations.

#### 7.6 Equilibrium Problems and Nonlinear Dynamical Systems

Nonlinear dynamical systems of the form

$$x_{n+1} = T x_n \cdot x_{n+1}$$

often have transition laws  $T$  that behave contractively only under a nonlinear, non-symmetric, or polyhedral distance.

For example:

- systems with switching dynamics,
- hybrid automata,
- piecewise linear flows,
- control systems with saturation.

Our results imply:

- asymptotic stability of equilibria,
- uniqueness of steady states,
- robustness to nonlinear distortions of the metric.

This unifies many prior results which required restrictive convexity or smoothness assumptions.

**8. Conclusion :** In this paper, we developed a unified framework for generalized  $\phi$ -contractions on polyhedral and nonlinear metric spaces. Our results extend the classical principles of Banach, Kannan, Chatterjee, Zamfirescu, and Proinov by showing that these well-known contraction types arise naturally as special cases of a broader geometric scheme. By incorporating polyhedral structures and nonlinear distance interactions, we demonstrated that fixed point theorems can be established without relying on linear or smooth geometries. This significantly enhances the applicability of fixed point theory to optimization, nonlinear dynamical systems, discrete structures such as graphs, and non-Euclidean metric spaces.

The framework presented here also suggests several promising directions for future research, including stability analyses under polyhedral perturbations, extensions to probabilistic or partial metrics, and the study of iterative schemes shaped by nonlinear metric behaviors. Overall, our work contributes a compact yet powerful generalization of contraction theory, offering new analytical tools and a deeper geometric understanding of fixed point phenomena in modern nonlinear analysis.

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